Solution
Math 220
HW # 11
December 7, 2018

Exercise 1. *Let* $A = \{0, 1, 2, 3\}$ *and let* $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$ *be a relation on* A*. Find the transitive closure of* R*.*

Solution.

$$R^t = R \cup \{(0, 3), (1, 0), (3, 1), (3, 2), (3, 3), (0, 0), (1, 2)\}.$$

□

Exercise 2. *Suppose* R *and* S *are binary relations on a set* A*.*

(a) *If* R *and* S *are reflexive, is* $R \cap S$ *reflexive? Why?*
(b) *If* R *and* S *are symmetric, is* $R \cap S$ *symmetric? Why?*
(c) *If* R *and* S *are transitive, is* $R \cap S$ *transitive? Why?*

Proof.

(a) Let $x \in A$. Since R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Therefore $(x, x) \in R \cap S$, and so $R \cap S$ is reflexive.

(b) Suppose $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Since R and S are symmetric, it follows that $(y, x) \in R$ and $(y, x) \in S$, hence $(y, x) \in R \cap S$. Therefore $R \cap S$ is symmetric.

(c) Suppose $(x, y), (y, z) \in R \cap S$. Then $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. Since R and S are both transitive, it follows that $(x, z) \in R$ and $(x, z) \in S$, hence $(x, z) \in R \cap S$. Therefore $R \cap S$ is transitive.

□

Exercise 3. *Suppose* R *and* S *are binary relations on a set* A*.*

(a) *If* R *and* S *are reflexive, is* $R \cup S$ *reflexive? Why?*
(b) *If* R *and* S *are symmetric, is* $R \cup S$ *symmetric? Why?*
(c) *If* R *and* S *are transitive, is* $R \cup S$ *transitive? Why?*

Proof.

(a) Let $x \in A$. Since R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Therefore $(x, x) \in R \cup S$, and so $R \cup S$ is reflexive.

(b) Suppose $(x, y) \in R \cup S$. Then $(x, y) \in R$ or $(x, y) \in S$. Without loss of generality, assume $(x, y) \in R$. Then it follows that $(y, x) \in R$ since R is symmetric, hence $(y, x) \in R \cup S$. Therefore $R \cup S$ is symmetric.

(c) $R \cup S$ is not transitive. Suppose $A = \{1, 2, 3\}$, $R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$, and $S = \{(1, 3), (3, 1), (1, 1), (3, 3)\}$. This R and S are transitive. Then $(2, 1), (1, 3) \in R \cup S$, but $(2, 3) \notin R \cup S$ since it is not in R or S. Therefore $R \cup S$ is not transitive.

$\square$

Exercise 4. *Define the relation S on $\mathbb{R}$ by xSy if and only if $x - y$ is an integer. Show that S is an equivalence relation.*

Proof.

Reflexive: Let $x \in \mathbb{R}$. Then since $x - x = 0$ is an integer, xSx. Therefore S is reflexive.

Symmetric: Let $x, y \in \mathbb{R}$ and suppose xSy. Then there is an integer k such that $x - y = k$. Then $y - x = -(x - y) = -k$, hence $y \sim x$.

Transitive: Let $x, y, z \in \mathbb{R}$ and suppose xSy and ySz. Then there are integers $k, l \in \mathbb{Z}$ such that $x - y = k$ and $y - z = l$. Adding these expressions gives

$$(x - y) + (y - z) = x - z = k + l.$$

Thus $x - z$ is an integer and xSz.

Therefore S is an equivalence relation. $\square$

Exercise 5. *Consider the following partition of the set $\{0, 1, 2, 3, 4\}$:*

$$\mathcal{P} = \{\{0, 2\}, \{1\}, \{3, 4\}\}.$$

What is the relation R that is induced by this partition? (Give R as a set of ordered pairs.)

Solution.

$$R = \{(0, 0), (0, 2), (2, 0), (2, 2), (1, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}.$$

$\square$

Exercise 6. *Let $A = \{1, 2, 3, 4, ..., 20\}$. An equivalence relation, $\sim$, is defined on A by*

$$x \sim y \text{ if and only if } 4|(x - y).$$

Find the distinct equivalence classes of $\sim$.

Solution. Begin with finding the equivalence class of 1. It consists of all numbers whose difference with 1 is divisible by 4.

$$[1] = \{1, 5, 9, 13, 17\}.$$

Since 2 isn't in this equivalence class, let's find it's equivalence class.

$$[2] = \{2, 6, 10, 14, 18\}.$$

3 hasn't been in an equivalence class so far, so find it's equivalence class.

$$[3] = \{3, 7, 11, 15, 19\}.$$

Next, we need 4's equivalence class.

$$[4] = \{4, 8, 12, 16, 20\}.$$

Since every number above 4 is already in one of the above equivalence classes, [1], [2], [3], and [4] are the distinct equivalence classes. $\square$

Exercise 7. *Let* $\sim$ *be the relation of congruence modulo* 3*. Which of the following equivalence classes are equal?*

$$[7], [-4], [-6], [17], [4], [27], [19].$$

Make sure to say why they are equal!

Solution. Since $7 - 4 = 3$ is divisible by 3, we know $7 \sim 4$, so $[7] = [4]$. Since $7 - 19 = -12$ is divisible by 3, we have $7 \sim 19$, so $[7] = [19]$.
Since $-4 - 17 = -21$ is divisible by 3, we have $-4 \sim 17$, so $[-4] = [17]$.
Since $-6 - 27 = -33$ which is divisible by 3, we have $-6 \sim 27$, so $[-6] = [27]$.

So we have $[7] = [4] = [19]$, $[-4] = [17]$, and $[-6] = [27]$. $\square$

Exercise 8.

(*a*) *Create an addition and multiplication table for* $\mathbb{Z}_5$.
(*b*) *Create an addition and multiplication table for* $\mathbb{Z}_6$.
(*c*) *In the multiplication tables, there is one big difference between* $\mathbb{Z}_5$ *and* $\mathbb{Z}_6$*, what is that difference?*

Solution.

(*a*) For $\mathbb{Z}_5$, the addition table is

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

and the multiplication table is

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(b) For $\mathbb{Z}_6$, the addition table is

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

and the multiplication table is

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(c) There are multiple things you could point out here, but the biggest deal is that in $\mathbb{Z}_6$, we can mulitply two nonzero numbers and get 0 as the product, whereas that does not happen in $\mathbb{Z}_5$. The numbers 2, 3, and 4 are called *zero divisors* in $\mathbb{Z}_6$.

$\square$